Theorem

Given a window function $w(t)$, define the time and frequency bandwidth as:

$$Time\ BW : \sigma_t(w) = \int t^2 |w(t)|^2 dt$$
$$Freq\ BW : \sigma_u(w) = \int u^2 |W(u)|^2 du$$

Then,

$$\sigma_t(w) \cdot \sigma_u(w) \geq \frac{1}{16\pi^2}$$

Proof

$$\sigma_t(w) \cdot \sigma_u(w) = \left[\int t^2 |w(t)|^2 dt\right] \left[\int u^2 |W(u)|^2 du\right]$$
$$= \left[\int t^2 |w(t)|^2 dt\right] \left[\int \left|\frac{1}{j2\pi} \frac{d}{dt} w(t)\right|^2 dt\right]$$
$$= \frac{1}{4\pi^2} \|tw(t)\|^2 \cdot \|\frac{dw(t)}{dt}\|^2$$

Explanation:

(1)

$$\frac{1}{j2\pi} \frac{d}{dt} x(t) \longleftrightarrow uX(u)$$

(2)

Norm is defined as:

$$\|x\| = \sqrt{\langle x, x \rangle}$$
$$\Rightarrow \|x\|^2 = \langle x, x \rangle$$

Where $\langle x, x \rangle$ is the inner product, and:

$$a(t) \longleftrightarrow A(u)$$
$$b(t) \longleftrightarrow B(u)$$
$$\langle a, b \rangle = \int a(t) b^*(t) dt$$
$$\langle a, b \rangle = \langle A, B \rangle$$

Where b^* is the conjugation vector of b.

Now back to the proof...

$$
\begin{aligned}
\sigma_t(w)\cdot\sigma_u(w) &= \frac{1}{4\pi^2}\|tw(t)\|^2 \cdot \|\frac{dw(t)}{dt}\|^2 \\
&\geq \frac{1}{4\pi^2}\left[\int tw(t)\frac{dw(t)}{\cancel{dt}}\cancel{dt}\right]^2 \\
&= \frac{1}{4\pi^2}\left[\int_{-\infty}^{+\infty} t\cdot d\left(\frac{w^2(t)}{2}\right)\right]^2 \\
&= \frac{1}{4\pi^2}\left[\frac{tw^2(t)}{2}\Big|_{-\infty}^{+\infty} - \int \frac{w^2(t)}{2}dt\right] \\
&= \frac{1}{16\pi^2}
\end{aligned}
$$

Explanation:

(3)

Cauchy–Schwarz Inequality:

$$\langle a, b\rangle = \|a\|\|b\|cos\theta \leq \|a\|\|b\|$$

(4)

Chain Rule:

$$
\begin{aligned}
\int d(ab) &= \int adb + \int bda \\
\Rightarrow \int adb &= \int d(ab) - \int bda \\
&= ab - \int bda
\end{aligned}
$$

(5)

We have some assumptions of w(t). First, w(t) is unit normalized. Second, w(t) is centered at zero. Third, w(t) decays fast enough.

$$\frac{tw^2(t)}{2}\Big|_{-\infty}^{+\infty} = 0$$

As w(t) decays fast enough (faster than $\sqrt{t}$), this term is zero.

(6)

$$\int \frac{w^2(t)}{2}dt = \frac{1}{2}$$

As w(t) is unit normalized:

$$\int |w(t)|^2 dt = \|w(t)\| = 1$$