Derivation of VINS-Mono IMU Pre-integration

IMU 原始观测:

Rotation from world → body_t

真值

$$\hat{a}_t = a_t + b_{a_t} + R^t_w g^w + n_a$$

noise

加速度观测 bias gravity

$$\hat{\omega}_t = \omega_t + b_{w_t} + n_w$$

角速度观测

assume $n_a \sim \mathcal{N}(0, \sigma_a^2)$ $n_w \sim \mathcal{N}(0, \sigma_w^2)$ 布朗运动

bias b_{a_t}, b_{w_t} ~ random walk ⇒ derivatives are Gaussian

$$\dot{b}_{a_t} = n_{b_a}, \quad \dot{b}_{w_t} = n_{b_w}$$

world 下, b_{k+1} 帧的位置

$$p^w_{b_{k+1}} = p^w_{b_k} + v^w_{b_k} \Delta t_k + \iint_{t \in [t_k, t_{k+1}]} \left(R^w_t (\hat{a}_t - b_{a_t} - n_a) - g^w \right) dt^2$$

b_k 位置 b_k 速度

$$v^w_{b_{k+1}} = v^w_{b_k} + \int_{t \in [t_k, t_{k+1}]} \left(R^w_t (\hat{a}_t - b_{a_t} - n_a) - g^w \right) dt$$

Note:

姿态 quaternion q:

$$\begin{array}{c} q_w \\ \\ q_v \\ \\ \end{array} \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \vec{n} \cdot \sin\frac{\theta}{2} \end{bmatrix}$$

θ: 旋转角度

$\vec{n}$: 旋转轴单位向量

四元数乘法:

$$\underset{4\times1}{p} \otimes \underset{4\times1}{q} \rightarrow 4\times1 = \begin{bmatrix} p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 \\ p_1q_0 + p_0q_1 + p_2q_3 - p_3q_2 \\ \vdots \end{bmatrix}$$

$$\begin{aligned} p \star q &= (p_0, p_1, p_2, p_3) \star (q_0, q_1, q_2, q_3) \\ &= \begin{bmatrix} p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 \\ p_1q_0 + p_0q_1 + p_2q_3 - p_3q_2 \\ p_2q_0 + p_0q_2 + p_3q_1 - p_1q_3 \\ p_3q_0 + p_0q_3 + p_1q_2 - p_2q_1 \end{bmatrix} \\ &= (p_0q_0 - \mathbf{p} \cdot \mathbf{q},\ p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}) \end{aligned}$$

或矩阵形式：

$$p\otimes q = [p]_L \cdot q_2 = [q]_R \cdot p$$

左乘矩阵　　右乘矩阵

$$q = \begin{bmatrix} q_w \\ \vec{q}_v \end{bmatrix}_{4\times 1}$$

$$[q]_L = q_w I + \begin{bmatrix} 0 & -q_v^T \\ q_v & [q_v]_\times \end{bmatrix}$$

$$[q]_R = q_w I + \begin{bmatrix} 0 & -q_v^T \\ q_v & -[q_v]_\times \end{bmatrix}$$

其中，$[\;]_\times$ 为反对称矩阵

$$[\vec{a}]_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

note: $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

$$[a]_\times b = \vec{a}\times\vec{b}$$ 等价于向量叉乘

$$q^w_{b_{k+1}} = q^w_{b_k} \otimes \int ? \, dt$$

旋转关于时间的导数？

是角速度，但具体是什么形式？
注意这里是向量微积分

旋转求导：

$$\dot{q}_{w_t} = \lim_{\Delta t\to 0} \frac{q_{w_{t+\Delta t}} - q_{w_t}}{\Delta t}$$ （定义求导）

$$= \lim_{\Delta t\to 0} \frac{q_{w_t}\otimes q_{t\to t+\Delta t} - q_{w_t}}{\Delta t}$$

$\Delta t\to 0$, $q_{t\to\Delta t} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \vec{n}\sin\frac{\theta}{2} \end{bmatrix} \xrightarrow{\theta\to 0} \begin{bmatrix} 1 \\ \vec{n}\cdot\frac{\theta}{2} \end{bmatrix}$

$$= \lim_{\Delta t\to 0} \frac{q_{w_t}\otimes \begin{bmatrix} 1 \\ \frac{\theta}{2}\vec{n} \end{bmatrix} - q_{w_t}\otimes \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix}}{\Delta t}$$

$$q_{w_t}\otimes \begin{bmatrix} 1 \\ \frac{\theta}{2}\vec{n} \end{bmatrix} = [\;]_R \cdot q_{w_t}$$

$$= q_w I + \underbrace{\begin{bmatrix} 0 & -q_v^T \\ q_v & -[q_v]_\times \end{bmatrix}}$$ 记为 $\Omega(q_v)$

$$= \lim_{\Delta t \to 0} \frac{(1 \cdot \cancel{I} + \Omega(\frac{\theta}{2}\vec{n})) \cdot q_{w_t} - (1 \cdot \cancel{I} + \Omega(\vec{0}))q_{w_t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[\Omega(\frac{\theta}{2}\vec{n}) - \cancel{\Omega(\vec{0})}]\, q_{w_t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \Omega\left(\frac{\theta \vec{n}}{2\Delta t}\right) q_{w_t}$$

$\lim_{\Delta t \to 0} \frac{\vec{n} \cdot \theta}{\Delta t} = \vec{\omega}$ 角速度

$$= \frac{1}{2}\Omega(\vec{\omega}) \cdot q_{w_t}$$

Ω 和论文定义不一致，因为 $q = \begin{bmatrix} q_v \\ q_w \end{bmatrix}$ 顺序相反

$$q^{w}_{b_{k+1}} = q^{w}_{b_k} \otimes \int_{t \in [t_k, t_{k+1}]} \frac{1}{2}\Omega(\hat{\omega}_t - b_{w_t} - n_w) q^{b_k}_t \, dt$$

计算需要 b_k 时刻状态，在优化过程中，一旦这些状态更新，则需重新计算积分，效率很低 ⇒ 预积分

Note: $\int 1\, dx = x$

$\iint 1\, dx^2 = \int x\, dx = \frac{1}{2}x^2$

变换坐标到 b_k local 坐标下，左右同乘 $R^{b_k}_w$

$$R^{b_k}_w \cdot p^w_{b_{k+1}} = R^{b_k}_w \cdot (p^w_{b_k} + v^w_{b_k}\Delta t_k - \frac{1}{2} g^w \Delta t_k^2) + \underbrace{\iint_t \underbrace{R^{b_k}_w \cdot R^w_t}_{R^{b_k}_t} \cdot (\hat{a}_t - b_{a_t} - n_a)\, dt^2}_{\alpha^{b_k}_{b_{k+1}}}$$

$$R^{b_k}_w \cdot v^w_{b_{k+1}} = R^{b_k}_w (v^w_{b_k} - g^w \Delta t_k) + \underbrace{\int R^{b_k}_t (\hat{a}_t - b_{a_t} - n_a)\, dt}_{\beta^{b_k}_{b_{k+1}}}$$

$\gamma^{b_k}_t = q^{b_k}_w \otimes q^w_t$

$$q^{b_k}_w \otimes q^w_{b_{k+1}} = \cancel{q^{b_k}_w} \otimes \cancel{q^w_{b_k}} \otimes \underbrace{\int_t \frac{1}{2}\Omega(\hat{\omega}_t - b_{w_t} - n_w)\, q^{b_k}_t\, dt}_{\gamma^{b_k}_{b_{k+1}}}$$

α, β, γ 只和 IMU 测量值有关

在离散时间中，可用 Euler / 中值积分. Paper 中用 Euler 作说明，代码中使用中值积分.

Note:

数值积分：给微分方程 $y' = f(t, y(t))$，求 Δt 时刻后的 y

$$y_{n+1} = y_n + \Delta t\, k_n \approx y(t_0) + \int_{t_0}^{t_0+\Delta t} y'(t)\, dt$$

Euler: $k_n = y'(t_n) = f(t_n, y_n)$

中值: $k_n = y'(t_n + \frac{1}{2}\Delta t)$

$$= f\left(t_n + \frac{\Delta t}{2}, \underbrace{y\left(t_n + \frac{\Delta t}{2}\right)}_{\text{未知，用 Euler 近似}}\right)$$

$$y\left(t_n + \frac{\Delta t}{2}\right) = y_n + \frac{1}{2}\Delta t \cdot f(t_n, y_n)$$

$$= f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{1}{2}\Delta t \cdot f(t_n, y_n)\right)$$

离散时间：$\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ 传播: 可看作无重力空间的位置、速度、姿态

初始 $\alpha_{b_k}^{b_k}$, $\beta_{b_k}^{b_k} = 0$; $\gamma_{b_k}^{b_k}$ identity ; n_a, n_w unkown, set as 0.

i is discrete moment $\in [t_k, t_{k+1}]$

δt is the time interval between $i+1$ and i.

$$\hat{\gamma}_{i+1}^{b_k} = \hat{\gamma}_i^{b_k} \otimes \hat{\gamma}_{i+1}^{i}$$

$$= \gamma_i^{b_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}(\hat{\omega}_i - b_{w_i})\,\delta t \end{bmatrix}$$

$\Delta t \cdot y'(t)$

$$\beta_{i+1}^{b_k} = \beta_i^{b_k} + \underbrace{R_i^{b_k}}_{R(\hat{\gamma}_i^{b_k})}(\underbrace{\hat{a}_i}_{\text{加速度读数}} - b_{a_i}) \cdot \delta t$$

无重力时的速度

$$\alpha^{b_k}_{i+1} = \alpha^{b_k}_{i} + \beta^{b_k}_{i}\delta t + \frac{1}{2}R(\hat{\gamma}^{b_k}_{i})(\hat{a}_t - b_{a_i})\delta t^2$$ 无重力时的位置.

Covariance Propagation

$\gamma^{b_k}_t$ 用4个数描述3Dof旋转，过参数化，因此用小扰动表示. around its mean ($\vec{0}$)

$$\gamma^{b_k}_t \approx \hat{\gamma}^{b_k}_t \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta^{b_k}_t \end{bmatrix}, \quad \delta\theta^{b_k}_t \to 0.$$

连续时间误差一阶导:

$$\delta\dot{\alpha}^{b_k}_t = \delta\beta^{b_k}_t$$ 位置误差的导数 = 速度误差

$$\delta\dot{b}_{a_t} = n_{b_a}$$ 零偏的导数 → 随机游走

$$\delta\dot{b}_{w_t} = n_{b_w}$$

速度: $\dot{\beta} = \hat{R}^{b_k}_t(a - \hat{b}_a)$

$$(\dot{\beta} + \Delta\dot{\beta}) = R\cdot\underbrace{\Delta R}\cdot(\hat{a} - n_a - (b_a + \Delta b_a))$$

$$\dot{\beta} + \Delta\dot{\beta}$$

对应 $\delta\theta \in so(3)$ 李代数 指数映射 十四讲 P_{79}

$\Delta R \in SO(3)$ 李群

$$\exp([\theta a]_\times)$$

$$\Delta R = I + [\delta\theta]_\times$$

$$= \cos\theta I + (1-\cos\theta)aa^T + \sin\theta[a]_\times$$

$$R(\hat{a} - b_a) + \Delta\dot{\beta} = R\cdot(I + [\delta\theta]_\times)(\hat{a} - n_a - b_a - \delta b_a)$$

$$\cancel{R(\hat{a} - b_a)} + \Delta\dot{\beta} = R(\cancel{\hat{a}} - n_a - \cancel{b_a} - \delta b_a) + R\cdot[\underset{\Delta}{\delta\theta}]_\times(\hat{a} - \underset{\Delta}{n_a} - b_a - \underset{\Delta}{\delta b_a})$$

$$\Delta\dot{\beta} = R(-n_a - \delta b_a) + R\underbrace{[\delta\theta]_\times}\underbrace{(\hat{a} - b_a)}$$

Δ: 一阶小量

相乘得二阶小量, 忽略.

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$

$$= R(-n_a - \delta b_a) - R(\hat{a} - b_a)_\times\delta\theta$$

note: $\dot{q} = \frac{1}{2}\Omega(\omega) q$

$$(\gamma \otimes \delta\gamma)^{\cdot} = \dot{\hat{\gamma}} = \frac{1}{2}\underline{\Omega(\omega)}\hat{\gamma}$$

相当于 矩阵乘法 求导

右乘矩阵 写回四元数乘法

$$= \frac{1}{2}\underline{\hat{\gamma}} \otimes (\hat{\omega} - \hat{b}_\omega - n_\omega)$$

$$\dot{\gamma} \otimes \delta\gamma + \gamma \otimes \delta\dot{\gamma} = \frac{1}{2}(\underline{\gamma \otimes \delta\gamma})(\hat{\omega} - n_\omega - (b_\omega + \delta b_\omega))$$

$$= \frac{1}{2}\Omega(\omega)\gamma = \frac{1}{2}\gamma \otimes (\hat{\omega} - b_\omega)$$

$$\frac{1}{2}\gamma \otimes (\hat{\omega} - b_\omega) \otimes \delta\gamma + \gamma \otimes \delta\dot{\gamma} = \frac{1}{2}(\gamma \otimes \delta\gamma) \otimes (\hat{\omega} - n_\omega - b_\omega - \delta b_\omega)$$

令 $\omega_1 = \hat{\omega} - b_\omega$, $\omega_2 = \hat{\omega} - n_\omega - b_g - \delta b_g$

$$\cancel{\tfrac{1}{2}\gamma} \otimes \omega_1 \otimes \delta\gamma + \overset{2}{\cancel{\gamma}} \otimes \delta\dot{\gamma} = \cancel{\tfrac{1}{2}\gamma} \otimes \delta\gamma \otimes \omega_2$$

$$\underline{\underline{2}}\,\delta\dot{\gamma} = \delta\gamma \otimes \omega_2 - \omega_1 \otimes \delta\gamma$$

$$= \Omega_R(\omega_2)\delta\gamma - \Omega_L(\omega_1)\delta\gamma$$

已知, $\delta\gamma = \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta \end{bmatrix}$

$$\Rightarrow \delta\dot{\gamma} = \begin{bmatrix} 0 \\ \frac{1}{\underline{\underline{2}}}\delta\dot{\theta} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \delta\dot{\theta} \end{bmatrix} = [\Omega_R(\omega_2) - \Omega_L(\omega_1)]\delta\gamma$$

$$= \begin{bmatrix} 0 & (\omega_1 - \omega_2)^T \\ (\omega_2 - \omega_1) & -(\omega_1 + \omega_2)_\times \end{bmatrix}\delta\gamma$$

略

$$\Rightarrow \delta\dot{\theta} = -(\hat{\omega} - b_\omega)_\times \delta\theta - \delta b_\omega - n_\omega$$

$$\begin{bmatrix} \delta\dot{\alpha} \\ \delta\dot{\beta} \\ \delta\dot{\theta} \\ \delta\dot{b}_a \\ \delta\dot{b}_\omega \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta\theta \\ \delta b_a \\ \delta b_\omega \end{bmatrix} + \begin{bmatrix} \quad \end{bmatrix} \cdot \begin{bmatrix} n_a \\ n_\omega \\ n_{b_a} \\ n_{b_\omega} \end{bmatrix}$$

$$= F_t\, \delta z_t^{b_k} + G_t n_t$$

已有一阶导，故 cov:

$$P^{b_k}_{t+\delta t} = (I + F_t \delta t) P^{b_k}_t (I + F_t \delta t)^T + (G_t \delta t) Q (G_t \delta t)^T$$

Kalman Filter
参看概率机器人

Jacobian:

$$J_{t+\delta t} = (I + F_t \delta t) J_t$$?

First order approximation:

$$\alpha^{b_k}_{b_{k+1}} = \hat{\alpha}^{b_k}_{b_{k+1}} + J^{\alpha}_{b_a} \delta b_{a_k} + J^{\alpha}_{b_w} \delta b_{w_k}$$

bias 估计值更新时的修正

离散时间：从 $\delta x_t \rightarrow \delta x_{t+1}$，利用中值积分

$$\delta x_{t+\delta t} = (I + F_t \delta t) \delta x_t + G_t n_t \delta t$$ ← $y_{n+1} = y_n + \delta t \cdot y'_n$

例：
$$\dot{\delta\theta} = -(\hat{\omega}_t - b_{w_t})_\times \delta\theta - \delta b_{w_t} - n_w$$ （连续时间）

$$\dot{\delta\theta} = -\left(\frac{\omega_t + \omega_{t+\delta t}}{2} - b_{w_t}\right)_\times \cdot \delta\theta - \delta b_{w_t} - \frac{n_{w_t} + n_{w_{t+\delta t}}}{2}$$

$$\delta\theta^{b_k}_{t+1} = \delta\theta^{b_k}_t - \delta t\left(\frac{\omega_t + \omega_{t+1}}{2} - b_{w_t}\right) \delta\theta^{b_k}_t - \delta b_{w_t} - \frac{n_{w_t} + n_{w_{t+1}}}{2} \delta t$$

其余 $\delta\beta$，$\delta\alpha$ 略.